

# Chiral Condensate, Master Field and all that in $QCD_2(N \rightarrow \infty)$ .

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## Abstract

We discuss the various aspects of two-dimensional  $QCD_2(N \rightarrow \infty)$  (the 't Hooft model[1]). Our main interest (motivated by the corresponding analysis in the four dimensional QCD) is the vacuum structure of the theory. We use the very general methods in the analysis, such as dispersion relations and duality in order to relate the known spectrum of  $QCD_2$  to the different vacuum characteristics.

We explicitly calculate (in terms of physical parameters like masses and matrix elements) the chiral condensate as well as the mixed vacuum condensates:

$$\langle 0 | \bar{q} (g \epsilon_{\mu\nu} G_{\mu\nu})^n q | 0 \rangle \sim M_{eff}^{2n} \langle 0 | \bar{q} q | 0 \rangle.$$

We interpret the factorization property for the mixed vacuum condensates as a reminiscent of the master field at large  $N$ .

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# 1. Introduction

In this paper we analyze the vacuum properties of two-dimensional QCD. The standard way of doing so is the “canonical” (or non-canonical) gauge fixing, analysis of different constraints, elimination of redundant degrees of freedom e t.c. We take the opposite approach. We assume that the spectrum is known and our main goal is find out the vacuum properties which would correspond to the given spectrum.

Since the pioneering paper by 't Hooft[1] in 1974,  $QCD_2(N \rightarrow \infty)$  has been subject of many investigations [2] - [7]. The list of problems discussed in these papers is impressive: scaling and  $e^+e^-$  annihilation; deep-inelastic scattering and Drell-Yan formula; Regge behavior, analysis of form factor and wide angle behavior of exclusive amplitudes; light cone quantization and e t.c..

The next step which has been taken is the analysis of the vacuum properties which would correspond to the well established spectrum. It has been realized that the vacuum of the theory in the 't Hooft limit

$$g^2 N \sim const. \quad N \rightarrow \infty, \quad m_q \gg g \sim \frac{1}{\sqrt{N}} \quad (1)$$

is quite nontrivial. In particular, the chiral condensate  $\langle \bar{q}q \rangle = -N \sqrt{\frac{g^2 N}{12\pi}}$  in the limit  $m_q \rightarrow 0$  has been calculated [8]<sup>3</sup>. The result was confirmed by numerical [9],[10] and independent analytical calculations[11]. Moreover, the method has been generalized for the nonzero quark mass and the corresponding explicit formula for the chiral condensate  $\langle \bar{q}q \rangle$  with arbitrary  $m_q$  has been obtained [12]. Let us note that there is no contradiction with the Coleman theorem [13] at this point, because the BKT (Berezinski-Kosterlitz-Thouless) behavior takes place in the large  $N$  limit[8].

Recently there has been a renewal of interest in the study of  $QCD_2$  [14]-[29]. The motivation for this interest was quite different for different people: it ranged from an attempt to find a string representation of QCD in four dimensions to mastering the instantons and master field in this theory. The important lesson to be learned from the new development, we believe, can be formulated as follows: the structure of gauge theories even in two dimensions (with and without matter fields) is very complicated. However, some physical questions can be formulated irrespective of number of dimensions. In certain circumstances, the correctly formulated questions can even be answered. This gives some hope (and hints) that the similar formulation of the analogous problems hopefully can be found in four dimensions.

This paper is largely motivated by analysis of nonperturbative wave functions with a minimal number of constituents in four dimensional QCD. As is known, such a function gives parametrically leading contributions to hard

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<sup>3</sup>The chiral limit  $m_q \rightarrow 0$  can be considered only after the limit  $N \rightarrow \infty$  is taken. These limits do not commute, see the next section for explanation.

exclusive processes. The corresponding wave functions within QCD have been introduced into the theory in the late seventies and early eighties [34] to describe the exclusive processes. We refer to the review papers [35],[36] on this subject for the detail definitions and discussions in the given context of Wilson operator expansion.

The two-dimensional  $QCD_2(N \rightarrow \infty)$  is the **ideal toy model** for such kind of problems, because the only physical states in the model are states with minimal number of constituents (the  $\bar{q}q$  mesons)<sup>4</sup>. The properties of these, nonperturbative wave functions might be quite nontrivial and unexpected. It would be very interesting to check the analytical nonperturbative methods (mainly the QCD sum rules) which have been developed for such an analysis.

As is known, the QCD sum rules approach operates with objects like vacuum condensates of different local operators. Eventually, if we knew all types of vacuum condensates, we would calculate an arbitrary correlation function and thus, through the dispersion relation, we would find the spectrum and amplitudes.

The subject of the present paper is analysis of the vacuum structure. More specific, we are interested in the exact calculation of the chiral, mixed vacuum condensates  $\langle 0|\bar{q}(g\epsilon_{\mu\nu}G_{\mu\nu})^n q|0\rangle$ . Once these nonperturbative condensates are found, than some more complicated nonlocal vacuum expectation values (like Wilson line  $\langle 0|W(x)|0\rangle \equiv \langle 0|\bar{q}(x)e^{ig\int_0^x A_\mu dx_\mu} q(0)|0\rangle$  and its superpositions  $\langle 0|W(x_1)W(x_2)W(x_3)\dots|0\rangle$ ) can be in principle evaluated.

The paper is organized as follows. In Sect.2 we review some  $QCD_2(N \rightarrow \infty)$  properties with emphasis on the vacuum structure of the theory. We review the spectrum found by 't Hooft and explain why this spectrum **unavoidably leads** to the existence of the chiral condensate  $\langle \bar{q}q \rangle$ . Through the dispersion relation we explicitly express the condensate in terms of spectrum and physical matrix elements. We qualitatively explain why the nonzero magnitude for  $\langle \bar{q}q \rangle$  does not contradict to the Coleman theorem and why the chiral limit  $m_q \rightarrow 0$  can be considered only after the large  $N$  limit is taken. In different words, we demonstrate that to preserve the 't Hooft solution (which corresponds to the selection only planar, leading at large  $N$ , diagrams) we need to require that the inequality  $m_q \gg g \sim 1/\sqrt{N}$  is fulfilled. Otherwise, the nonplanar diagrams which might have the singular factors like  $1/m_q$  come into the game and destroy the 't Hooft solution.

Sect.3 is devoted to the calculation of the mixed vacuum condensates. We try to avoid, in all cases, the discussion of such complicated problems as infrared and ultraviolet regularization, the problem of splitting of two local operators, and many other problems which unavoidably accompany any calculation in local field theory. Instead, we use the knowledge of existence of the nonperturbative vacuum condensate  $\langle \bar{q}q \rangle$  to calculate the more complicated, higher dimensional condensates<sup>5</sup>. The **crucial reason** for such incredible

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<sup>4</sup> Let us remind that an additional creation of  $\bar{q}q$  pairs is suppressed by a factor  $1/N$ .

<sup>5</sup> The gluon in  $QCD_2$  is not a physical degree of freedom. Thus, one could anticipate

simplification is the factorization property  $\langle Q_1 \cdot Q_2 \rangle = \langle Q_1 \rangle \langle Q_2 \rangle + 0(1/N)$  of vacuum expectation values in the large  $N$  limit. Thus, all vacuum condensates can be reduced to  $\langle \bar{q}q \rangle$  and its powers. We believe that if the master field is found, it should reproduce the vacuum properties mentioned above.

Sect.4 is our conclusion and outlook. To be more specific, we discuss some phenomenological consequences of the obtained formulae which might be interesting for the analysis of nonperturbative wave functions and heavy-light quark system.

## 2. The spectrum and chiral condensate in $QCD_2(N \rightarrow \infty)$ .

The model we shall consider consists of quark in fundamental representation interacting via an  $SU(N)$  color gauge group. We follow the notation of ref.[2] and present the 't Hooft equation [1] in the following form:

$$m_n^2 \phi_n(x) = \frac{m_q^2}{x(1-x)} \phi_n(x) - m_0^2 P \int dy \frac{\phi_n(y)}{(x-y)^2}, \quad (2)$$

where symbol  $P$  notes as the principal value of the integral, and  $0 < x < 1$  is the fraction of the total momentum of the bound state carried by quark  $q$  with mass  $m_q$ . The quantity  $m_0^2 \equiv \frac{g^2 N}{\pi}$  is the basic mass scale in the theory and the index  $n$  classifies the ordering number of the bound states  $|n, p\rangle$  with total momentum  $p_\mu$ . The same wave function can be expressed in terms of the following matrix element [6]:

$$\phi_n(x) = \sqrt{\frac{N}{\pi}} \int dy_+ e^{-iy_+(1-2x)p_-} \langle 0 | \bar{q}(-y) q(y) | n, p \rangle |_{y_- = 0}. \quad (3)$$

Let us note that matrix element on the right hand side is written in the light cone gauge  $A_- = 0$ ; to restore the manifest gauge invariance one can insert the standard exponential factor  $e^{ig \int A_- dy_+}$  into the formula (3).

Let us review some important properties of equation (2). The entire spectrum is discrete and classified by the integer number  $n$ . The wave functions  $\phi_n(x)$  are orthogonal, complete and obey the following boundary conditions

$$\phi_n(x) \rightarrow [x(1-x)]^\beta, \quad x \rightarrow 0, \quad x \rightarrow 1, \quad \pi\beta \cot(\pi\beta) = 1 - \frac{m_q^2}{m_0^2}. \quad (4)$$

It turns out the spectrum of states is almost linear for large  $n$ :

$$m_n^2 \simeq \pi^2 m_0^2 n, \quad \phi_n(x) \simeq \sqrt{2} \sin(\pi n x) \quad (5)$$

and does not depend on mass of the quark. What is more important, in the chiral limit  $m_q \rightarrow 0$  the lowest level (we call it  $\pi$  meson) tends to zero  $m_\pi^2 \sim m_q$  and one could expect the nonzero magnitude for the chiral condensate.

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that any local operator can be expressed in terms of the quark fields. As is expected, this is indeed the case.

We define the chiral condensate in the current algebra terms as follows:

$$0 = \lim_{p_\mu \rightarrow 0} i \int d^2x e^{ipx} \partial_\mu \langle 0 | T \{ \bar{q} \gamma_\mu \gamma_5 q(x), \bar{q} \gamma_5 q(0) \} | 0 \rangle = \quad (6)$$

$$2i \langle 0 | \bar{q} q | 0 \rangle + 2m_q \cdot \langle 0 | T \{ \bar{q} i \gamma_5 q(x), \bar{q} i \gamma_5 q(0) \} | 0 \rangle.$$

As we already mentioned, the only states of 't Hooft's solution are the quark-antiquark bound states. Thus, they must saturate the dispersion relation. Upon inserting this complete set of mesons to the (6) one thus obtains:

$$\langle 0 | \bar{q} q | 0 \rangle = -m_q \sum_n \frac{N}{\pi} \frac{f_n^2}{m_n^2}, \quad (7)$$

where  $f_n$  is defined in terms of the following matrix elements

$$\langle 0 | \bar{q} i \gamma_5 q | n \rangle = \sqrt{\frac{N}{\pi}} f_n, \quad f_n = \frac{m_q}{2} \cdot \int_0^1 \frac{\phi_n(x)}{x(1-x)} dx \quad (8)$$

In the chiral limit the only state which can contribute to the formula (7) is the  $\pi$  meson. Its matrix element can be calculated exactly and we end up with the following expression for the chiral condensate in the  $m_q \rightarrow 0$  limit [8]:

$$\langle 0 | \bar{q} q | 0 \rangle = -N \frac{m_0}{\sqrt{12}}, \quad m_0^2 = \frac{g^2 N}{\pi}, \quad f_{n=0} = \frac{m_0 \pi}{\sqrt{3}}, \quad m_\pi^2 = m_q \cdot \frac{2m_0}{\sqrt{3}}. \quad (9)$$

As was expected, we find that  $\langle 0 | \bar{q} q | 0 \rangle \sim N$ . Besides that, as we already noticed in [8], if we put  $m_q = 0$  from the very beginning, then  $\langle 0 | \bar{q} q | 0 \rangle = 0$ . This corresponds to the different regime when  $m_q \ll g \sim 1/\sqrt{N}$ , when nonplanar diagrams come into the game. We discuss this point a little bit later. The last remark is the observation that the entire nonzero answer for the condensate comes from the infrared region of the integration in eq.(8):  $x \sim 0, x \sim 1$  which corresponds to the situation when one of the quarks carries all the momentum and the second one is at rest.

The sum (7) can be calculated exactly for arbitrary  $m_q$  [12]. The crucial point is that for arbitrary  $m_q$  the nonzero contribution comes from the highly excited states ( $n \gg 1$ ) only. The properties of these states are well-known:

$$f_n^2 \rightarrow \pi^2 m_0^2, \quad m_n^2 \rightarrow \pi^2 m_0^2 \cdot n, \quad n \gg 1, \quad (10)$$

and thus the sum (7) can be explicitly evaluated with the result [12]:

$$\langle 0 | \bar{q} q | 0 \rangle = \frac{m_q N}{2\pi} \{ \log(\pi\alpha) - 1 - \gamma_E + (1 - \frac{1}{\alpha}) [I(\alpha) - \alpha I(\alpha) - \log 4] \}, \quad (11)$$

where  $\alpha = \frac{m_0^2}{m_q^2}$ ,  $\gamma_E = 0.5772..$  is Euler's constant and

$$I(\alpha) = \int_0^\infty \frac{dy}{y^2} \frac{1 - \frac{y}{\sinh y \cosh y}}{[\alpha(y \coth y - 1) + 1]}.$$

This result is exact for large  $N$  and arbitrary quark mass as far as we remain in the 't Hooft regime (1), i.e.  $m_q \gg g \sim 1/\sqrt{N}$ . It reduces to the eq.(9) in the limit  $\alpha \rightarrow \infty$ , as it should.

The last condition ( $m_q \gg g$ ) which has to be satisfied for the 't Hooft solution to be valid, requires some additional explanation. Roughly speaking, nonplanar diagrams may contain a factor  $\sim m_q^{-1}$  which at  $m_q = 0$  blow up and the theory changes completely. The concept of the proof that there exists a factor  $\sim m_q^{-1}$  in nonplanar diagrams is the following.

Let us consider the correlation function for  $p \rightarrow 0$

$$i \int d^2x e^{ipx} \langle 0 | T \{ \bar{q}q(x), \bar{q}q(0) \} | 0 \rangle = P(p^2) \quad (12)$$

The 't Hooft solution suggests that only planar graphs are taken into account and, consequently, the spectral density contains only the contribution of one meson states. For these contributions  $P_{planar} \sim N$ . In the chiral limit, we can calculate the two-pion contribution exactly! This contribution is **not accounted for** in deriving (2). Of course, the two-pion contribution is suppressed by a factor  $1/N$ . However, it contains a term  $\sim \frac{m_0^2}{m_\pi^2}$  which tends to infinity for  $m_q \rightarrow 0$ . The presence of the factor  $\sim m_q^{-1}$  in nonplanar diagrams leads to the aforementioned constraint on  $m_q$  (1).

Now, let us explicitly demonstrate the existence of the term  $\sim m_q^{-1}$  for the two-pion contribution. In order to do so, let us write down a dispersion relation for  $P$ :

$$P(0) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds}{s} ImP(s), \quad (13)$$

where  $ImP(s)$  is the physical spectral density. The  $\pi\pi$  contribution is fixed uniquely by (9), because of the special role of pions [8]:

$$\begin{aligned} \langle \pi\pi | \bar{q}q | 0 \rangle |_{p \rightarrow 0} &= \frac{m_0\pi}{\sqrt{3}}, \quad \frac{1}{\pi} ImP^{\pi\pi}(s) = \frac{m_0^2\pi^2}{6} \frac{1}{\sqrt{s(s-4m_\pi^2)}}, \\ P^{\pi\pi}(0) &= \frac{m_0^2\pi^2}{6} \int_{4m_\pi^2}^{\infty} \frac{ds}{s\sqrt{s(s-4m_\pi^2)}} = \frac{m_0^2\pi^2}{12m_\pi^2} \sim \frac{1}{m_q}. \end{aligned} \quad (14)$$

It is clear that the only cause for a singular  $\sim 1/m_q$  behavior is the finiteness of the pion matrix elements at zero momentum. At the same time this contribution does not contain the large factor  $N$  which accompany a one meson contribution to the same correlator. To suppress these nonplanar diagrams we need to require  $N \gg \frac{m_0^2}{m_\pi^2}$ . Thus, we would expect that some kind of phase transition may occur in the region  $m_q \sim g$ , where we would expect a complete restructuring of the theory.

The last subject we would like to discuss in this section is the strict Coleman theorem [13] which states that a continuous symmetry cannot be

broken spontaneously in two dimensional theories. As we discussed earlier [8] we expect that as in the  $SU(N \rightarrow \infty)$  Thirring model (where the chiral symmetry is “almost” spontaneously broken [37]), the BKT effect [38] operates in regime (1). This fact also confirms the ’t Hooft spectrum: states with opposite  $P$  parity are not degenerate in mass and there is an “almost” Goldstone boson with  $m_\pi^2 \sim m_q + 1/N$ .

To be more specific, one can show [8] that in  $QCD_2(N \rightarrow \infty)$  the behavior of the proper two-point correlation function is as follows:

$$\langle 0|T\{\bar{q}_L q_R(x), \bar{q}_R q_L(0)\}|0\rangle \sim x^{-\frac{1}{N}}. \quad (15)$$

Such a behavior together with cluster property at  $x \rightarrow \infty$  implies the existence of the condensate at  $N = \infty$  in a full agreement with our previous discussion. At the same time, for any finite but large  $N$ , the correlator falls off very slowly demonstrating the BKT-behavior with no signs of contradiction to the Coleman theorem.

### 3. The vacuum expectation value of the mixed vacuum condensates.

We start from the consideration of the simplest mixed vacuum expectation value

$$\langle \bar{q} D_\mu D_\nu q \rangle = \frac{1}{2} g_{\mu\nu} \langle \bar{q} D_\lambda D_\sigma (\gamma_\lambda \gamma_\sigma - \epsilon_{\lambda\sigma} \gamma_5) q \rangle = \frac{1}{4} g_{\mu\nu} \langle \bar{q} i g \epsilon_{\lambda\sigma} G_{\lambda\sigma} \gamma_5 q \rangle, \quad (16)$$

where  $[D_\mu, D_\nu] = -igG_{\mu\nu}$  is the field strength tensor of the gauge field and we have used the well-known properties of the  $\gamma_\mu$  matrices in two dimensions. Thus, in the chiral limit,  $m_q \rightarrow 0$ , the operator we are interested in can be expressed exclusively in terms of the field strength tensor  $G_{\mu\nu}$ .

What’s more, in two dimensions an arbitrary operator can be reduced to the form which contains the field strength tensor  $ig\epsilon_{\mu\nu}G_{\mu\nu}$  only. Indeed, the covariant derivatives  $\langle \bar{q} D^n \cdot D_\mu D_\mu q \rangle$  placed at the very right and at the very left (near the quark fields) can be transformed into the operator  $ig\epsilon_{\mu\nu}G_{\mu\nu}$  as before. To do the same thing with operators  $D_\mu$  which placed somewhere in the middle, we need to act (say) on the right until the quark field is reached. By doing so, step by step, we create many additional terms which are either: commutator like  $[D_\mu, D_\nu] = -igG_{\mu\nu}$  which is the field strength operator or commutators like  $\sim [D_\lambda, \epsilon_{\mu\nu}G_{\mu\nu}]$ . Fortunately, in two dimensions these terms are related to creation of the quark- antiquark fields and we discard them in according to large  $N$  counting rule <sup>6</sup>.

Indeed, the quark-antiquark operator comes from the equation of motion  $D_\mu G_{\mu\nu} \sim \bar{q} \gamma_\nu q$  when we sum up over the common index  $\mu$ . This is true in any number of dimensions. In two dimensional case, even without a common index  $\mu$ , the action of  $D_\mu$  on field strength tensor produces the quark-antiquark

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<sup>6</sup> Of course this is not the case in four dimensions where  $[D_\mu^2, G_{\lambda\sigma}]$  is independent operator which can not be reduced to some quark fields.

operator. This can be seen by noting that in 2 dimensions  $G_{\mu\nu}(x) \sim \epsilon_{\mu\nu} E(x)$  with scalar function  $E \sim \epsilon_{\lambda\sigma} G_{\lambda\sigma}$  and  $D_\mu E \sim \bar{q} \gamma_\mu \gamma_5 q$ .

Thus, we end up with the vacuum condensates  $\langle \bar{q}(gE)^n q \rangle$  which are expressed exclusively in terms of the field strength tensor and our nearest problem is their calculation.

Here we sketch the idea of this calculation. We choose the light cone gauge,  $A_- = \frac{1}{\sqrt{2}}(A_0 - A_1) = 0$ . In this gauge we have the usual constraint in the gauge sector (Gauss law) (for more details see e.g. [11]):

$$\partial_- E^{ab} \sim g(q_+^{\dagger b} q_+^a - \frac{1}{N} \delta^{ab} q_+^{\dagger c} q_+^c),$$

where  $a, b$  are the color indices. Here the right moving fermion  $q_+$  are dynamical degrees of freedom; the left-moving fermion  $q_-$  are non-dynamical degrees of freedom in this gauge. The latter can be eliminated by the following constraint:

$$\partial_- q_- \sim m_q q_+.$$

The next step is to use the Gauss law to calculate the mixed vacuum condensate. For the simplest case it looks as follows:

$$\langle \bar{q} i g \epsilon_{\mu\nu} G_{\mu\nu} \gamma_5 q \rangle \sim \langle q_-^{\dagger a} g E^{ab} q_+^b \rangle \sim \langle q_-^{\dagger a} g^2 \frac{1}{\partial_-} (q_+^a q_+^{\dagger b}) q_+^b \rangle$$

Although the explicit expression of the Green function  $\frac{1}{\partial_-}$  is well known (it is the step function  $\epsilon(x_- - x'_-)$  in the coordinate space, see e.g. [11]), as will be clear soon, we do not need its manifest form.

What does matter, is the important property of the large  $N$  limit that reduces the expectation value of a product of any invariant operators to their factorized values[39]. Thus, the condensate under consideration can be presented in the following form

$$\langle \bar{q} i g \epsilon_{\mu\nu} G_{\mu\nu} \gamma_5 q(0) \rangle \sim g^2 \langle q_-^{\dagger a}(0) q_+^a(x'_-) \rangle \cdot \langle \frac{1}{\partial_-} q_+^{\dagger b} q_+^b(0) \rangle$$

The first term on the right hand side of this expression is reduced to the chiral condensate  $\langle \bar{q} q \rangle$ . Indeed, the additional terms like  $\frac{x_-^n}{n!} \langle \bar{q} (\partial^-)^n q \rangle$  that come from the Taylor expansion, are zero because of the Lorentz invariance of the vacuum state. The second term is also reduced to the chiral condensate if one takes into account the constraint mentioned above and which we would like to write down in the following way:  $\frac{1}{\partial_-} q_+ \sim \frac{1}{m_q} q_-$ .

As we have already mentioned, neither explicit formula for the Green function  $\frac{1}{\partial_-}$  has been used, nor specific regularization has been assumed in this derivation.

The final formula for the simplest mixed condensate (16) which accounts for all numerical factors takes the following form:

$$\frac{1}{2!} \langle \bar{q} (x_\mu D_\mu)^2 q \rangle = -\frac{1}{8} x^2 \frac{g^2 \cdot \langle \bar{q} q \rangle^2}{m_q}. \quad (17)$$



Few comments are in order. First of all, as we expected, the higher dimensional condensate can be expressed in terms of the fundamental chiral condensate (9,11). We note also that the mixed condensate (17) has the same  $N$  dependence as the fundamental one,  $\langle \bar{q}q \rangle$  (remember,  $g^2 \langle \bar{q}q \rangle \sim g^2 N \sim 1$ ).

The sign of the right hand side in eq.(17) is the result of the calculation. We would like to pause here to compare this result with somewhat analogous in four- dimensional QCD, where the corresponding formula looks as follows:

$$\frac{1}{2!} \langle \bar{q}(x_\mu D_\mu)^2 q \rangle = \frac{1}{16} x^2 \cdot \langle \bar{q} i g \sigma_{\mu\nu} G_{\mu\nu}^a \frac{\lambda^a}{2} q \rangle \simeq \frac{1}{16} x^2 0.8 \text{GeV}^2 \langle \bar{q}q \rangle. \quad (18)$$

In both cases, the sign of the ratio

$$\frac{\langle \bar{q}(x_\mu D_\mu)^2 q \rangle}{\langle \bar{q}q \rangle} \sim x^2 \frac{\langle \bar{q} D_\mu^2 q \rangle}{\langle \bar{q}q \rangle} \sim x^2 < 0$$

is negative for negative  $x^2$ , where operator expansion is effective. In four dimensions this sign has very deep physical meaning, because there is one-to-one correspondence between such kind of the vacuum condensates and mean values of transverse moments for the pion wave function[40]<sup>7</sup>. There is no such interpretation in 2d (no transverse direction in this case), however, one can argue that for the space-like interval, the sign in the eq.(17) does correspond to the positivity of the Hermitian operator  $(iD_1)^2$ .

The next step is the application of the same procedure for the operators with arbitrary number of gluon field insertion  $E^{ab}$ . It can be done in the same way as before just because of the factorizability mentioned above. Thus we arrive to the following formula

$$\langle \bar{q}(D_\mu D_\mu)^n q \rangle = \frac{1}{2^n} \langle \bar{q}(i g \epsilon_{\lambda\sigma} G_{\lambda\sigma} \gamma_5)^n q \rangle = \left( -\frac{g^2 \langle \bar{q}q \rangle}{2m_q} \right)^n \langle \bar{q}q \rangle, \quad (19)$$

which is our main result. The comment to this formula can be formulated in the following way. Each time the insertion of an additional factor  $(D_\mu D_\mu)$ , which is proportional to the field strength tensor  $gE$ , gives one and the same numerical factor (19). Situation can be interpreted as we would have a **classical** master field [39] which we insert in place of  $gE$  in the vacuum condensates. Because of its classicality, it gives one and the same numerical factor.

The second important msg. is as follows. The vacuum condensate of an arbitrary local operator can be reduced through the equation of motion and constraints to the fundamental quark condensate (9,11). This is exactly what one could expect in  $QCD_2$  where a gluon is not a physical degree of freedom, but rather is constrained auxiliary field which can and should be expressed in terms of quark fields.

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<sup>7</sup>This is not a big surprise, however, because the pion is Goldstone particle and its matrix elements very often can be reduced to the vacuum condensates through the PCAC.

With these remarks in mind one could calculate (in principle) the vacuum expectation value of an arbitrary nonlocal operator like string operator  $\langle W(x) \rangle$  and its superpositions  $\langle W(x_1), W(x_2), W(x_3) \dots \rangle$ . The only what we need to do is to use the Taylor expansion

$$\langle W \rangle = \langle 0 | \bar{q}(x) P e^{ig \int_0^x A_\mu dx_\mu} q(0) | 0 \rangle = \sum_{n=0}^{n=\infty} \frac{1}{(2n)!} \langle \bar{q}(x_\mu D_\mu)^{2n} q \rangle \quad (20)$$

for the nonlocal operator we are interested in and substitute the values for the vacuum condensates we calculated previously (19). However we better stop here.

## 5. Conclusion and Outlook.

We have calculated the mixed vacuum condensates (19) and have explained of how to reduce an arbitrary local vacuum condensate to already known values (19). We also have demonstrated of how to calculate the vacuum expectation value of the nonlocal operators (like string operator and its superpositions) by mean the Taylor expansion and reducing the problem to the previous one (19). We shall not discuss here the theoretical issue of related problems (which are quite interesting, by the way). Instead, in this conclusion we would like to mention some “phenomenological” applications which might be interesting for the future investigation.

First of all, as we already noticed,  $\langle W \rangle$  naturally appears in the analysis of the heavy-light quark system  $\bar{q}Q$  within operator product expansion. Indeed, if we consider along with [41],[42] the correlation function  $\langle T \{ \bar{q}Q(x), \bar{Q}q(0) \} \rangle$ , describing this system, we end up (in the limit  $M_Q \rightarrow \infty$ ) with the object which completely factorized (in accordance with HQET[43]) from the heavy quark and which was called NLC (nonlocal condensate)[42]:

$$\langle T \{ \bar{q}Q(x), \bar{Q}q(0) \} \rangle \sim \langle \bar{q}(x) P \exp(ig \int_0^x A_\mu dx_\mu) q(0) \rangle + \text{perturb. part.} \quad (21)$$

All nontrivial, large distance physics of the system is hidden there. It has been noticed recently that this system might provide a definition of the constituent quark in QCD. Together with perturbative contribution one should expect the following behavior for this correlator[42]:

$$\langle T \{ \bar{q}Q(x), \bar{Q}q(0) \} \rangle \sim e^{-\Lambda \cdot x}. \quad (22)$$

We do not expect that the function  $\langle W \rangle$  can provide such a behavior by itself in t’Hooft model. The reason for that is related to the perturbative terms, proportional to  $\sim (\frac{g^2 N}{\pi})^n (x^2)^n$ . These contributions go on the same foot as nonperturbative ones due to the dimensional coupling constant  $g$  in 2d, and they interfere with expansion (20).

Naively, one could expect that nothing like that might happen in four dimensional  $QCD_4$ , where the coupling constant is dimensionless and no

power corrections might occur in perturbative series. However this is not completely true, because of the so-called renormalons, which may provide some effective power corrections [44].

Another, but related issue is as follows. As is known, [45] the analysis of different problems (like inclusive decays, distribution function...) in heavy-light quark system, requires the precise information about the corresponding behavior in the so-called end-point domain. Formally, it requires the knowledge of the infinite set of matrix elements like  $\langle H_Q | \bar{Q} D_{\mu_1} \dots D_{\mu_n} Q | H_Q \rangle$ . Our remark is that such matrix elements can be explicitly calculated in  $QCD_2$  as we discussed before. We believe that  $QCD_2$  as a toy model may provide at least a hint on analytical properties of this, so-called universal distribution function  $F(y)$ . This universal distribution function can be presented as matrix element  $F(y) \sim \int e^{ixy} \langle H_Q | \bar{Q}(x) e^{ig \int_0^x A_\mu dx_\mu} Q(0) | H_Q \rangle$ . Operator expansion presents this (presumably regular function  $F(y)$ ) as a series of delta function and its derivatives  $F(y) \sim \sum a_n \delta^n(1-y)$ . Thus, to find out the analytical properties of such a function in the real world is a difficult task which requires the knowledge of distant terms of the series. The lessons which might be provided by  $QCD_2$  could be useful.

At the end, then, we come back to the beginning. We have already noticed in the Introduction that  $QCD_2$  is the nice model for analysis of nonperturbative wave functions themselves and methods which one can use to extract the corresponding information. We refer the reader to the recent papers on this subject [46],[47] for the references and new development. Here we would like to note that in  $QCD_2$  we know the nonperturbative wave functions as well as whole set of local condensates (19) which play the crucial role in this study. Thus, one may try to check the methods which have been developed for corresponding analysis.

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